Optimized Dynamical Decoupling Sequences in Protecting Two-Qubit States

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Aperiodic dynamical decoupling (DD) sequences of π pulses are of great interest to decoherence control and have been recently extended from single-qubit to two-qubit systems. If the environmental noise power spectrum is made available, then one may further optimize aperiodic DD sequences to reach higher efficiency of decoherence suppression than known universal schemes. This possibility is investigated in this work for the protection of two-qubit states, using an exactly solvable pure dephasing model including both local and nonlocal noise. The performance of optimized DD sequences in protecting two-qubit states is compared with that achieved by nested Uhrig's DD (nested-UDD) sequences, for several different types of noise spectrum. Except for cases with noise spectrum decaying slowly in the high-frequency regime, optimized DD sequences with tens of control pulses can perform orders of magnitude better than that of nested-UDD. A two-qubit system with highly unbalanced local noise is also examined to shed more light on a recent experiment. Possible experiments that may be motivated by this work are discussed.

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I. INTRODUCTION

In one way or another any two-level system (qubit) is coupled to some environmental degrees of freedom. This inevitable system-environment coupling leads to decoherence. To protect quantum coherence as a key resource for quantum technologies, many schemes have been proposed to dynamically eliminate the unwanted qubit-environment coupling [1–5].

Analogous to the spin-echo technique widely adopted in nuclear-magnetic-resonance studies [6], various dynamical decoupling (DD) sequences of instantaneous control pulses [2, 7–9] have been extensively studied. In particular, due to Uhrig's DD (UDD) sequence [8, 9], research activities focusing on DD have surged recently [10–15]. In addition to its high efficiency in theory, the power of UDD lies in its universality [16, 17]. Indeed, the working mechanism of UDD does not rely on detailed assumptions about the system-environment coupling or about the environment. Nevertheless, if the actual form of the noise spectrum of the environment is available, then a locally optimized DD (LODD) [10] sequence based on the noise spectrum can outperform UDD, with the pulse locations optimized according to an exact decoherence function [18–20]. The reason for the success of optimization is simple. In suppressing the pure-dephasing of a qubit, a UDD sequence minimizes a decoherence filter function (defined later) in the neighborhood of zero frequency but gradually becomes less effective for large frequencies. As such, if the noise spectrum of the environment is known, then its actual behavior at appreciably nonzero frequencies makes room for further optimization of DD sequences. This optimization approach is somewhat in the same spirit of the continuous DD approach [3, 4], insofar as both attempt to make the full use of the noise spectrum.

Extension of DD to two-qubit (or even multi-qubit) systems is crucial towards efficient protection of quantum entanglement. For a known initial state of a two-qubit system, an extended UDD sequence is found by involving nonlocal control operators, with its theoretical performance essentially identical with that in one-qubit UDD cases [21]. For general situations, nested-UDD sequences, initially proposed for suppressing both dephasing and relaxation in one-qubit systems [22], are advocated to protect two-qubit states in a universal manner, i.e., without knowing the details of the system-environment coupling or of the noise spectrum [23, 24]. For example, it was shown that to lock an unknown superposition state of two known basis states to the Nth order, three layers of UDD sequences and hence about N^3 control pulses in total will be needed [21]. If the initial state is totally unknown, then to reach the same level of decoherence suppression one needs four layers of UDD sequences and hence about N^4 control pulses [23, 24]. One important question then arises. That is, if the noise spectrum of the environment of a two-qubit system is available, then can we significantly improve entanglement protection by further optimizing the locations of the instantaneous control pulses as what was done in [10, 18] for single-qubit systems? If yes, then the required number of pulses can be much less and more understandings of entanglement

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protection might emerge. Using an approach extended from the above-mentioned LODD for single-qubit systems, this question is answered here via a pure dephasing model of an open two-qubit system. The performance of optimized DD sequences in preserving two-qubit states is compared with that achieved by nested-UDD, for several different types of noise spectrum. Except for the Lorentzian type of noise spectrum, it is found that optimized DD sequences can protect two-qubit states orders of magnitude better than nested-UDD. As a result, on the one hand nested-UDD can be seen as a powerful and a general-purpose scheme for protecting two-qubit states, and on the other hand the optimized DD approach can be seen as system-specific DD schemes with even better performance. In addition, to shed more light on a recent experiment of entanglement protection via DD [25], a two-qubit system that shares important noise features with the experiment is studied.

Our plan for this paper is as follows. In the next section, we discuss a model that describes pure dephasing processes of a two-qubit system in the presence of instantaneous π pulses. Based on exact expressions of decoherence effects, we elaborate in Sec. III our optimization procedure to find the optimized pulse locations. In Sec. IV, the performance of optimized two-qubit DD sequence is illustrated in four subsections treating different types of noise spectrum. Sec. V concludes this work and proposes two types of experiments.

II. PURE DEPHASING MODEL OF TWO-QUBIT SYSTEMS

In general, decoherence as a complicated process involves both population relaxation and dephasing. Yet, if we increase the strength of an external polarization field such that all the energy level splittings are sufficiently large, then the relaxation can be made negligible within a time scale of interest and dephasing becomes the only source of decoherence. Under such a pure dephasing assumption, the environmental noise may be modeled as classical random fields causing random phase shifts, a valid treatment in many dephasing environments like spin bath in solid-state systems [10, 12, 13] and background noise in superconducting qubits [26]. Here we follow the methodology proposed in [9, 26]. We can then write the pure-dephasing Hamiltonian of an open two-qubit system as

$$H = f_1(t)\sigma_{z_1} + f_2(t)\sigma_{z_2} + f_3(t)\sigma_{z_1}\sigma_{z_2}, \tag{1}$$

where $f_i(t)$ are assumed to be independent random noise variables with a Gaussian distribution. We further assume

$$\langle f_i \rangle = 0, \tag{2}$$

$$\langle f_i(t_1) f_i(t_2) \rangle = g_i(t_1 - t_2), \tag{3}$$

with the correlation function $g_i(t)$ being an even function. The $f_1(t)$ $[f_2(t)]$ term in the Hamiltonian depicts the noise seen by the first (second) qubit alone; whereas the $f_3(t)$ term reflects how noise may jointly impact the two qubits. In the following we loosely call the $f_3(t)$ noise term as a term of nonlocal noise. If the two qubits are far apart, then this nonlocal noise term should be very small and can be neglected. As a consequence the decoherence control problem is expected to share important features with single-qubit DD [27] (note that $f_1(t)$ and $f_2(t)$ are assumed to be independent here). However, of our concern in this study are two nearby qubits interacting with each other, and therefore the nonlocal noise term should be included to account for random fluctuations in the qubit-qubit mutual interaction (for example, due to the fluctuations in qubit-qubit distance caused by lattice vibrations, or in the context of super-conducting qubits [28], due to the fluctuations in a third device that is responsible for qubit-qubit coupling). Certainly, in a more realistic environment the $f_3(t)$ term might be correlated with the local noise terms. Such kind of potential correlations are neglected in our model. From a different perspective, one may regard the first qubit as a part of the environment of the second qubit, and then the $f_3(t)$ term models how the dynamics of one qubit might change the environment experienced by the other qubit. Throughout we assume dimensionless units.

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be the eigenstates of σ_z, then a general two-qubit pure state at time zero can be written as

$$|\Psi(0)\rangle = \alpha|\downarrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle + \gamma|\uparrow\downarrow\rangle + \eta|\uparrow\uparrow\rangle, \tag{4}$$

where the upward or downward arrows in each of the four components represent the spin states of the two qubits. The Hamiltonian in Eq. (1) then gives rise to the following state vector at time t,

$$\begin{aligned} |\Psi(t)\rangle &= \alpha e^{-\mathrm{i}[-F_1(t)-F_2(t)+F_3(t)]}|\downarrow\downarrow\rangle \\ &+ \beta e^{-\mathrm{i}[-F_1(t)+F_2(t)-F_3(t)]}|\downarrow\uparrow\rangle \\ &+ \gamma e^{-\mathrm{i}[F_1(t)-F_2(t)-F_3(t)]}|\uparrow\downarrow\rangle \\ &+ \eta e^{-\mathrm{i}[F_1(t)+F_2(t)+F_3(t)]}|\uparrow\uparrow\rangle, \end{aligned} \tag{5}$$

where $F_i(t) \equiv \int_0^t f_i(t')dt'$. To analyze the density matrix $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ averaged over noise histories (denoted $\langle\cdot\rangle$), we further define four basis states $|0\rangle = |\downarrow\downarrow\rangle, |1\rangle = |\downarrow\uparrow\rangle, |2\rangle = |\uparrow\downarrow\rangle, |3\rangle = |\uparrow\uparrow\rangle$. Then all the averaged density matrix elements (in the absence of DD control pulses) can be easily worked out. For example, the mean value of $\rho_{01}(t)$, denoted $\bar{\rho}_{01}(t)$, can be expressed as

$$\begin{aligned} \bar{\rho}_{01}(t) &= \alpha^*\beta\langle e^{-\mathrm{i}[F_2(t)-F_3(t)]}e^{-\mathrm{i}[F_2(t)-F_3(t)]}\rangle \\ &= \alpha^*\beta e^{-2\langle[F_2(t)-F_3(t)]^2\rangle} \\ &= \alpha^*\beta e^{-2[\langle F_2^2(t)\rangle+\langle F_3^2(t)\rangle]}. \end{aligned} \tag{6}$$

Here, in obtaining the last equality we have used the relation $\langle f_2 f_3\rangle = 0$ as well as the Gaussian nature of the noise. Similar expressions can be obtained for all other density matrix elements.

In our pure-dephasing model, the error operator for the first (second) qubit is σ_{z_1} (σ_{z_2}) only, whose detrimental effect can be suppressed by a local control operator σ_{x_1} (σ_{x_2}). As seen from Refs. [23, 24], a nested-UDD scheme with two layers of σ_{x_1} and σ_{x_2} pulses can suppress the dephasing to the Nth order, with about N^2 pulses in total. To examine if we can further improve the performance by optimizing the pulse locations, let us now consider the following scenario: n pulses of π rotation along axis-x are applied to the first qubit, with the pulse locations given by $t_1, t_2, ..., t_n$, whereas m analogous pulses are applied to the second qubit at times $t_{1'}, t_{2'}, ..., t_{m'}$. As a result, totally $n+m$ pulses are applied to the two-qubit system. We arrange the $n+m$ pulse timings in increasing order and denote them by $t_{1''}, t_{2''}, ..., t_{(n+m)''}$, with $t_{j''} < t_{(j+1)''}$. At each of such instants, either the σ_{z_1} or σ_{z_2} operator switches its sign. At the same time, the operator $\sigma_{z_1}\sigma_{z_2}$ changes its sign $n+m$ times at these instants. This motivates us to define three switch functions [9]:

$$\begin{aligned}
s_1(t') &= (-1)^{k_1}, \quad t_{k_1} < t' \le t_{(k_1+1)}, \quad k_1 = 0, 1, ..., n, \\
s_2(t') &= (-1)^{k_2}, \quad t_{k_2'} < t' \le t_{(k_2+1)'}, \quad k_2 = 0, 1, ..., m, \\
s_3(t') &= (-1)^{k_3}, \quad t_{k_3''} < t' \le t_{(k_3+1)''}, \quad k_3 = 0, 1, ..., n+m,
\end{aligned}$$

with $t_0 = 0$, and $t_{n+1} = t_{(m+1)'} = t_{(n+m+1)''} = t$. For times outside the domain $[0, t]$ these switch functions are defined to be zero. The influence of the DD pulses can then be expressed in a rather compact form. Still taking the decay of $\bar{\rho}_{01}$ as an example, we obtain

$$\bar{\rho}_{01}(t) = \alpha^* \beta e^{-2[\langle \tilde{F}_2^2(t)\rangle + \langle \tilde{F}_3^2(t)\rangle]}, \tag{7}$$

where $\tilde{F}_i(t)$ is given by

$$\tilde{F}_i(t) = \int_{-\infty}^{\infty} f_i(t') s_i(t') dt'. \tag{8}$$

To proceed we next define three filter functions [9] from the Fourier transform of $s_i(t)$, i.e.,

$$\int_{-\infty}^{\infty} s_1(t') e^{\mathrm{i}\omega t'} dt' = \frac{\mathrm{i}}{\omega} y_n(\omega t), \tag{9}$$

$$\int_{-\infty}^{\infty} s_2(t') e^{\mathrm{i}\omega t'} dt' = \frac{\mathrm{i}}{\omega} y_m(\omega t), \tag{10}$$

$$\int_{-\infty}^{\infty} s_3(t') e^{\mathrm{i}\omega t'} dt' = \frac{\mathrm{i}}{\omega} y_{n+m}(\omega t). \tag{11}$$

Here, the filter function with M (which is n, m, or $n+m$) pulses is defined as

$$y_M(\omega t) = 1 + (-1)^{M+1} e^{\mathrm{i}\omega t} + 2\sum_{j=1}^{M} (-1)^j e^{\mathrm{i}\omega t \delta_j}, \tag{12}$$

with δ_j being the pulse location normalized to the total duration t, i.e., $\delta_j = t_j/t$. The decay factor in the exponential of Eq. (7) then reduces to

$$\begin{aligned}
&\langle \tilde{F}_2^2(t)\rangle + \langle \tilde{F}_3^2(t)\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 \; s_2(t_1) g_2(t_1 - t_2) s_2(t_2) \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2 \; s_3(t_1) g_3(t_1 - t_2) s_3(t_2) \\
&= \frac{1}{\pi}\int_0^{\infty} |y_m(\omega t)|^2 \frac{S_2(\omega)}{\omega^2} d\omega \\
&\quad + \frac{1}{\pi}\int_0^{\infty} |y_{m+n}(\omega t)|^2 \frac{S_3(\omega)}{\omega^2} d\omega,
\end{aligned} \tag{13}$$

where $S_i(\omega)$ is the noise spectrum (or spectral density), namely, the Fourier transform of the noise correlation function $g_i(t)$, with

$$g_i(t) = \frac{1}{\pi}\int_0^{\infty} S_i(\omega)\cos(\omega t) d\omega. \tag{14}$$

Note that an equivalent expression may be derived using an exactly solvable spin-boson model with pure-dephasing. Interestingly, the decay factor in Eq. (7) is seen to be the sum of two decoherence functions, each being analogous to that in the previous DD studies [8, 9]. Yet, the problem here is still rather complicated for two reasons. First, for the same control pulse, the coefficient $(-1)^j$ can take different values in the three filter functions. Indeed, its actual value depends on its relative pulse location in the time sequences of t_{k_1}, of $t_{k_2'}$, or of $t_{k_3''}$. Second, density matrix elements do not decay in the same fashion, and hence the decay of all off-diagonal density matrix elements needs to be accounted for.

Repeating the above procedure for all other density matrix elements, we finally obtain the averaged full density matrix $\bar{\rho}(t)$ as follows:

$$\begin{pmatrix}
|\alpha|^2 & \alpha^*\beta e^{-\Gamma_2(t)-\Gamma_3(t)} & \alpha^*\gamma e^{-\Gamma_1(t)-\Gamma_3(t)} & \alpha^*\eta e^{-\Gamma_2(t)-\Gamma_1(t)} \\
\beta^*\alpha e^{-\Gamma_2(t)-\Gamma_3(t)} & |\beta|^2 & \beta^*\gamma e^{-\Gamma_2(t)-\Gamma_1(t)} & \beta^*\eta e^{-\Gamma_1(t)-\Gamma_3(t)} \\
\gamma^*\alpha e^{-\Gamma_1(t)-\Gamma_3(t)} & \gamma^*\beta e^{-\Gamma_2(t)-\Gamma_1(t)} & |\gamma|^2 & \gamma^*\eta e^{-\Gamma_2(t)-\Gamma_3(t)} \\
\eta^*\alpha e^{-\Gamma_2(t)-\Gamma_1(t)} & \eta^*\beta e^{-\Gamma_1(t)-\Gamma_3(t)} & \eta^*\gamma e^{-\Gamma_2(t)-\Gamma_3(t)} & |\eta|^2
\end{pmatrix},$$

with the decay exponents for the off-diagonal elements given by

$$\Gamma_1 = \int_0^\infty |y_n|^2 \frac{S_1(\omega)}{\omega^2} d\omega, \quad (15)$$

$$\Gamma_2 = \int_0^\infty |y_m|^2 \frac{S_2(\omega)}{\omega^2} d\omega, \quad (16)$$

$$\Gamma_3 = \int_0^\infty |y_{n+m}|^2 \frac{S_3(\omega)}{\omega^2} d\omega. \quad (17)$$

For convenience, unimportant constants in Eqs. (15)-(17) are either set to unity (e.g., $t = 1$) or absorbed into the noise spectrum.

Before ending this section, we stress that our straightforward calculations above are made possible by first assuming the statistical independence of $f_1(t)$ and $f_2(t)$. If $f_1(t)$ and $f_2(t)$ has nonzero correlations (noise under this correlated situation may be also called nonlocal noise, which is much different from our case here), then many of the density matrix elements cannot be evaluated analytically.

III. OPTIMIZATION PROCEDURE

Next we aim to optimize the pulse locations to keep $\bar{\rho}(t)$ as close as possible to the initial state. Taking the trace fidelity $C(t) = \mathrm{Tr}[\bar{\rho}(t)\rho(0)]$ as a measure of DD performance, it is straightforward to carry out the optimization if the initial state is known. However, in many cases a two-qubit state to be protected or stored is unknown, and as such the average fidelity for all possible initial states can be of more interest. Averaging $C(t)$ over all initially pure states and using the fact that $\langle|\alpha|^2\rangle = \langle|\beta|^2\rangle = \langle|\gamma|^2\rangle = \langle|\eta|^2\rangle = \frac{1}{4}$, we arrive at

$$\bar{C}(t) = \frac{1}{4} + \frac{1}{4}(e^{-\Gamma_1-\Gamma_2} + e^{-\Gamma_1-\Gamma_3} + e^{-\Gamma_2-\Gamma_3}). \quad (18)$$

The optimization then becomes the minimization of $\Phi(t) \equiv 4[1 - C(t)]$. The function $\Phi(t)$, called the performance function below, is given by

$$\Phi(t) = 3 - (e^{-\Gamma_1-\Gamma_2} + e^{-\Gamma_1-\Gamma_3} + e^{-\Gamma_2-\Gamma_3}). \quad (19)$$

Clearly $0 \leq \Phi(t) \leq 3$, and a smaller value of $\Phi(t)$ indicates a higher degree of decoherence suppression. Therefore $\Phi(t)$ can be also regarded an error function.

To minimize the value of the performance function, we first consider a total of $n + m$ pulses, among which m pulses are applied to the second qubit. Assuming that the $n + m$ pulses are applied at the timings

$$\delta_1 < \delta_2 < \delta_3 < ... < \delta_{n+m},$$

one first needs to pick out m pulse locations to apply the σ_{x_2} control operator. The number of such choices is given by the combination C^m_{n+m}. Then the analytic form of the three filter functions $y_n(\omega)$, $y_m(\omega)$, and $y_{n+m}(\omega)$ can be fixed. We next minimize the performance function $\Phi(t)$ by optimizing the $(n+m)$ pulse locations using the line-search algorithm. In every iteration, the algorithm searches for the minimum along the direction set by a gradient. We use $n + m$ equally spacing pulses and a two-layer nested-UDD sequence as our initial guesses. Once the result converges, we obtain the locally optimized locations for $n + m$ control pulses in a particular $(n + m, m)$ scheme. To find the optimal pulse locations, we test all C^m_{n+m} possible choices of pulse allocations for the second qubit. Furthermore, by scanning the value of m from 1 to the total number of pulses, we can find an optimal pulse-number partition for a fixed total number of pulses.

TABLE I: Impact of nonlocal noise on the performance of optimized two-qubit DD sequences. The local noise spectrum is assume to be $S_1(\omega) = \omega\Theta(1 - \omega)$ for the first qubit and $S_2(\omega) = \omega\Theta(1 - \omega)$ for the second qubit. $n + m = 8$. In the column of pulse location, a listed integer j means that the jth pulse will be a σ_{x_2} pulse (i.e., applied to the second qubit). Performance is optimized by considering all C^2_8 and C^4_8 possible pulse allocations for the two qubits.

Nonlocal spectrum $S_3(\omega)$	Pulse location	Performance
$2\omega\Theta(2 - \omega)$	2,4,6,8	4.59×10^{-5}
$0.5\omega\Theta(0.5 - \omega)$	2,4,6,8	4.59×10^{-5}
$0.1\omega\Theta(0.1 - \omega)$	2,4,6,8	4.43×10^{-5}
$\frac{0.2}{\omega^2+1}$(infinite cutoff)	2,4,6,8	1.67×10^{-3}
no nonlocal noise	2,4,6,8	4.08×10^{-10}

IV. PERFORMANCE OF OPTIMIZED SEQUENCES

In the following, for convenience we set $t = 1$ when comparing cases of different noise spectrum. Effects of varying t can be understood as the result of a rescaling of the strength and shape of the noise spectrum [20]. To appreciate that the issue of two-qubit decoherence suppression is significantly different from a one-qubit problem, let us first illustrate the influence of the nonlocal noise term $f_3(t)\sigma_{z_1}\sigma_{z_2}$ on the performance of DD with $n + m = 8$. Computational examples are shown in Table I. It is seen that in the absence of nonlocal noise, the optimized performance (error) is of the order of 10^{-10}, and the optimized DD pulses are applied at

$$[0.10 \quad 0.10 \quad 0.35 \quad 0.35 \quad 0.65 \quad 0.65 \quad 0.90 \quad 0.90],$$

which is simply two optimized 4-pulse sequences simultaneously applied to the two qubits. However, this DD sequence cannot suppress any nonlocal noise because every pair of such simultaneous pulses will keep the sign of $\sigma_{z_1}\sigma_{z_2}$ unchanged, and the associated decay exponent Γ_3 would be the same as that without DD, which is given

TABLE II: Performance of optimized two-qubit DD sequences as compared with nested-UDD, for a few examples of local and nonlocal noise spectrum. Performance is optimized for each combination C^{m}_{n+m} taking into account all possibilities of allocating m pulses to the second qubit. In the column of pulse location, a listed integer j means that the jth pulse is a σ_{x_2} pulse (i.e., applied the second qubit) during the first half of the dynamics. The second half of the pulse sequence is symmetric to the first half. Ohmic noise spectrum with hard cutoff is considered.

$S_1 = \omega\Theta(1-\omega), S_2 = \omega\Theta(1-\omega), S_3 = 2\omega\Theta(2-\omega)$		
Pulse combination	Pulse location	Performance
nested-UDD(2)	3	7.32×10^{-4}
C^2_8	3	8.66×10^{-5}
C^4_8	2,4	4.59×10^{-5}
nested-UDD(3)	4,8	2.45×10^{-6}
C^3_{15}	4,8	3.04×10^{-7}
C^5_{15}	2,5,8	6.14×10^{-9}
C^9_{15}	1,3,5,7,8	1.17×10^{-10}
$S_1 = \omega\Theta(1-\omega), S_2 = \omega\Theta(1-\omega), S_3 = 0.5\omega\Theta(0.5-\omega)$		
Pulse combination	Pulse location	Performance
nested-UDD(2)	3	3.26×10^{-4}
C^2_8	3	8.14×10^{-5}
C^4_8	2,4	4.59×10^{-5}
nested-UDD(3)	4,8	1.66×10^{-6}
C^3_{15}	3,8	1.88×10^{-7}
C^5_{15}	3,5,8	7.06×10^{-11}
C^9_{15}	1,3,4,6,8	6.26×10^{-10}

by

$$\Gamma_3 = 4\int_0^{\infty} S_3(\omega)\frac{\sin^2(\omega/2)}{\omega^2}d\omega. \qquad (20)$$

Interestingly, as shown in Table I, even with a very weak nonlocal noise added, e.g., $S_3(\omega) = 0.1\omega\Theta(0.1-\omega)$, where $\Theta(\cdot)$ is the Heaviside step function, the minimal error that can be achieved by an optimized DD sequence is already increased by five orders of magnitude! This clearly addresses the importance of taking nonlocal noise into consideration for decoherence suppression. With this understanding we are now ready to examine the optimization of DD in two-qubit systems.

A. Spectrum with Hard Cutoff

Previous work showed that in single-qubit cases, an optimized DD sequence can greatly outperform UDD for an Ohmic noise spectrum with hard cutoff [10, 18–20]. It is observed that this finding also holds in our two-qubit dephasing model. Here, we set the local noise spectrum of the two qubits to be the same, and then vary the intensity of the nonlocal noise spectrum, from twice as much as the local noise spectrum to a relatively weak one. As seen from Table II, the optimized DD sequence is in general much better than nested-UDD. In particular, for a 15-pulse sequence ($n+m = 15$), the optimization improves the performance by many orders of magnitude as compared with a two-layer nested-UDD(3) (here nested-UDD(k) means that k pulses applied to the second qubit in the outside layer, and within each layer a k-pulse UDD sequence is applied to the first qubit). As a matter of fact, by comparing Table II and Table VI, it is seen that the best performance of optimized 15-pulse DD sequence for $S_3 = 0.5\omega\Theta(0.5-\omega)$ can even be two orders of magnitude better than nested-UDD(4), which requires 24 pulses. This indicates that if the noise spectrum is known, then the required pulse number to achieve a given degree of decoherence suppression does not have to scale with N^2 as in nested-UDD(N). Further, as suggested by the results in Table II, to achieve the best performance the number of σ_{x_2} pulses is larger than that suggested by nested-UDD. But interestingly, this does not necessarily mean that the number of σ_{x_2} pulses should be as close as possible to the number of σ_{x_1} pulses. For example, the C^5_{15} (instead of C^7_{15}) case shown in the bottom of Table II gives the best performance. This is one of the subtle consequences of nonlocal noise spectrum. Lastly, as $S_3(\omega)$ changes from $2\omega\Theta(2-\omega)$ to $0.5\omega\Theta(0.5-\omega)$, it is seen from Table II that not only the optimized performance changes, but also the whereabouts of σ_{x_2} pulses become much different. The physical intuition why the σ_{x_2} pulses should be shifted in this manner is far from obvious, reflecting the importance of the relative locations of σ_{x_2} pulses with respect to σ_{x_1} pulses.

In single-qubit cases [9, 18–20], LODD is more powerful for a noise spectrum with a larger cutoff. This motivated us to investigate some representatives cases with

TABLE III: Same as in Table II, but for Ohmic noise spectrum with hard cutoff at larger frequencies and for $1/f$ spectrum with hard cutoff.

$S_1 = \omega\Theta(5-\omega), S_2 = \omega\Theta(5-\omega), S_3 = \omega\Theta(3-\omega)$		
Pulse combination	Pulse location	Performance
nested-UDD(2)	3	1.55
C^2_8	3	0.80
C^4_8	2,4	0.54
nested-UDD(3)	4,8	0.36
C^3_{15}	3,8	6.63×10^{-2}
C^7_{15}	2,4,6,8	1.48×10^{-6}
$S_1 = \frac{1}{\omega}\Theta(10-\omega), S_2 = \frac{1}{\omega}\Theta(10-\omega), S_3 = \frac{1}{\omega}\Theta(5-\omega)$		
Pulse combination	Pulse location	Performance
nested-UDD(2)	3	0.61
C^2_8	3	0.60
C^4_8	2,4	0.41
nested-UDD(3)	4,8	0.32
C^3_{15}	4,8	0.22
C^7_{15}	2,4,6,8	9.96×10^{-5}

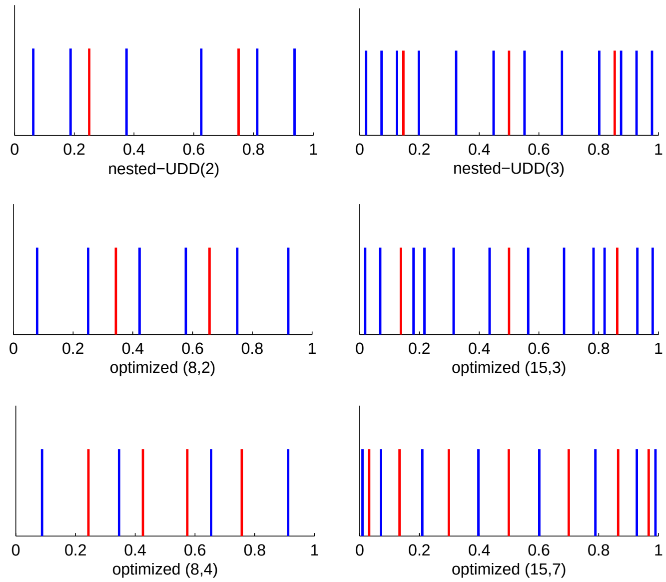

FIG. 1: Comparsion of specific pulse locations between nested-UDD and our optimized two-qubit DD sequences. $S_1 = \omega\Theta(5-\omega)$, $S_2 = \omega\Theta(5-\omega)$ and $S_3 = \omega\Theta(3-\omega)$. The left panel is for $n+m = 8$, and the right panel is for $n+m = 15$. The grey (red) lines stand for timings of the pulses applied to the second qubit.

a larger frequency cutoff. Here we consider Ohmic and $1/f$ spectrum in Table III (with cutoff). Here the $1/f$-type noise is of interest because it has been observed in many solid state implementations, especially in superconducting qubits [26, 29, 30]. As seen from Table III, in both cases the two-qubit dephasing can be greatly suppressed. Interestingly, although the Ohmic and the $1/f$ cases represent drastically different noise spectrum, in these two cases the optimized 15-pulse two-qubit DD sequences give similar performance that is five or six orders of magnitude better than nested-UDD(3). Clearly then, the improvement afforded by optimization is sensitive to the cutoff frequency value, but rather insensitive to the shape of the noise spectrum before the cutoff. Similar improvements have been observed previously in single qubit LODD [9, 18].

In Fig. 1 we show the optimized pulse locations in the case of $S_1 = \omega\Theta(5-\omega)$, $S_2 = \omega\Theta(5-\omega)$ and $S_3 = \omega\Theta(3-\omega)$. In the case of $n+m = 8$ with $m = 2$, the locations of all σ_{x_2} pulses in the optimized DD sequence differ remarkably from that in nested-UDD(2). In the case of $n+m = 15$ with $m = 3$, the first σ_{x_2} pulse is the third pulse in the optimized two-qubit sequence but the fourth pulse in nested-UDD(3). These observations might be relevant to future experiments.

B. Spectrum with Soft Cutoff

Noise spectrum with soft cutoff only gradually decays to zero. The associated task of decoherence suppression is more challenging. References [9, 18, 19, 26] showed that for single-qubit dephasing caused by soft-cutoff noise, UDD can only give a performance analogous to the Carr-Purcell-Meiboom-Gill (CPMG) sequence [31]. Mathematically, this can be understood from the decay exponents Γ_i determined by an integral of a filter function multiplied by a noise spectrum. Because the noise spectrum is not rapidly vanishing in the high frequency regime, the behavior of a filter function in the high frequency regime may be important and as a result the optimization becomes less effective.

Here we study the optimization of DD using various noise spectrum combinations, involving both super-Ohmic ($\omega^\alpha, \alpha > 1$) spectrum and Lorentzian spectrum with soft cutoff. The results are presented in Table IV. In the first super-Ohmic case with an exponential soft cutoff, the improvement of optimized DD sequence over nested-UDD is still magnificent (note that a non-optimized DD sequence might even speed up the decoherence process [32]). Indeed, the exponential cutoff can be regarded as an intermediate case between a truely slow cutoff (e.g., Lorentzian spectrum) and a hard cutoff studied above. The second case [$S_1(\omega) = S_2(\omega) = \omega\Theta(1-\omega)$, $S_3 = \frac{0.2}{\omega^2+1}$] should be compared with Table II, with the only difference being that a nonlocal Ohmic spectrum replaced by a weak nonlocal Lorentzian spectrum. Consequences of this change in the nonlocal noise spectrum are: (i) Performance of nested-UDD is greatly reduced, so is the performance of optimized DD, (ii) With the same value of $n+m$, optimized DD can only improve the performance over nested-UDD slightly, (iii) Finally, the performance still improves with the increase of $n+m$, but very slowly. Similar observations can be made from the third case in Table IV, where the local noise spectrum is assumed to be Lorentzian.

C. Asymmetric Local Spectrum

So far local noise spectrum is assumed to be the same for the two qubits. In practice the local noise spectrum of individual qubits can be much different. The recent experiment on DD of a two-qubit system represents one excellent example of unbalanced local noise spectrum [25]. In the experiment the dephasing time of a nuclear spin is two orders of magnitude larger than that of an electron spin. The DD pulses are then applied to the electron spin only. After a two-flip control sequence, the lifetime of a pseudo-entangled state in the experiment becomes analogous to that of the nuclear spin, which implies that control of the nuclear spin is needed to further improve the entanglement protection.

To mimic an environment like the experimental setup in Ref. [25], we consider an environment with strongly

TABLE IV: Same as in Table II, but involving noise spectrum without hard cutoff.

$S_1 = \omega^3 e^{-\omega^2}, S_2 = \omega^3 e^{-\omega^2}, S_3 = \omega e^{-\omega^2}$		
Pulse combination	Pulse location	Performance
nested-UDD(2)	3	5.31×10^{-3}
C_8^4	2,4	1.04×10^{-3}
nested-UDD(3)	4,8	1.44×10^{-4}
C_{15}^5	2,5,8	5.25×10^{-9}
$S_1 = \omega\Theta(1-\omega), S_2 = \omega\Theta(1-\omega), S_3 = \frac{0.2}{\omega^2+1}$		
Pulse combination	Pulse location	Performance
nested-UDD(2)	3	4.36×10^{-3}
C_8^4	2,4	1.67×10^{-3}
nested-UDD(3)	4,8	1.20×10^{-3}
C_{15}^9	2,4,5,7,8	4.74×10^{-4}
$S_1 = \frac{0.2}{\omega^2+1}, S_2 = \frac{0.2}{\omega^2+1}, S_3 = \omega\Theta(1-\omega)$		
Pulse combination	Pulse location	Performance
nested-UDD(2)	3	2.87×10^{-2}
C_8^4	2,4	2.08×10^{-2}
nested-UDD(3)	4,8	1.36×10^{-2}
C_{15}^7	2,4,6,8	3.96×10^{-3}

TABLE V: Same as in Table II, but for an example where the local environments of the two qubits are highly unbalanced.

$S_1 = 10\Theta(10-\omega), S_2 = 0.1\Theta(0.1-\omega), S_3 = 0.05\Theta(0.05-\omega)$		
Pulse combination	Pulse location	Performance
C_4^0	no pulse	1.30
C_4^2	2	2.00
C_8^0	no pulse	2.00×10^{-2}
C_8^2	3	7.64×10^{-3}
C_8^4	1,3	1.30
C_8^6	1,2,4	2.00
C_{12}^0	no pulse	1.99×10^{-2}
C_{12}^2	4	1.57×10^{-7}
C_{12}^4	3,5	6.25×10^{-6}
C_{12}^6	2,4,6	7.45×10^{-3}
C_{12}^8	1,3,4,6	1.29

asymmetric local noise. For convenience the noise spectrum is assumed to be a constant subject to a hard cutoff, i.e., $S_1(\omega) = 10\Theta(10-\omega)$, $S_2(\omega) = 0.1\Theta(0.1-\omega)$, and $S_3(\omega) = 0.05\Theta(0.05-\Omega)$. For these noise parameters the dimensionless dephasing rate of the first qubit is about two orders of magnitude larger than that of the second qubit. The optimization results are shown in Table V. At least two interesting observations can be made. First, in the case of $n+m = 4$, assigning some control pulses to the second qubit may bring down the performance. Hence, if the local noise spectrum is strongly unbalanced and if $n+m$ is highly limited, then it is beneficial to control only one of the two qubits. This is consistent with the experimental implementation in Ref. [25]. Second, when the minimized total error approaches the error caused by the free decay of the second qubit, solely applying σ_{x_1} pulses can no longer improve the DD performance. As shown in Table V, the DD performance without any σ_{x_2} pulses is bounded by about 10^{-2}. However, if now two σ_{x_2} pulses are applied, then the DD performance can be improved by many orders of magnitude (e.g., in the 12-pulse case). Note also that for fixed $n+m$, increasing m too much undermines the performance. Again, this is because the number of pulses applied to the two qubits should be balanced so that errors of the two qubits are suppressed to the same level. Less pulses should be applied to the second qubit because it is more weakly coupled to its environment. As such, the number of σ_{x_1} pulses (n) should always dominate. In Fig. 2 we display the optimized pulse locations in several cases. The left panels of Fig. 2 represent optimized pulse locations if $m = 0$. The right panels show the optimized pulse locations if $m = 2$. Interestingly, in the upper right panel, it is found that for $n+m = 4$, the two pulses for the second qubit should be applied at the same time, thus canceling themselves. Our optimization procedure therefore requests a null action on the second qubit if $n+m$ is small. For the case of optimized (8,2) shown in Fig. 2, the pulse combination is the same as in nested-UDD(2), but with optimized pulse timings. We have also compared the optimized (12,2) case in Fig. 2 with an asymmetric nested-UDD of total 11 pulses, namely, UDD(3) on the first qubit in the inner layer and UDD(2) on the second qubit in the outer layer. The performance of this asymmetric nested-UDD is only 0.517, still far from what is achieved here (only at the cost of one additional pulse in the middle interval of the inner layer).

D. Limitations on Optimized DD

As in single-qubit DD, we would naturally attempt to increase the total pulse number $n+m$ to achieve better and better DD performance. However, the optimization procedure faces much difficulty when $n+m$ increases considerably. First, the optimization algorithm becomes much slower in each run and may fail to give a converged result. For example, for $n+m = 24$, the converged pulse locations might be inappropriate, i.e., violating the initial ordering of the pulse locations. Worse still, for $n+m = 24$, we have to run the algorithm 924 times for the single combination C_{24}^{12} (12 pulses for the second qubit). Hence computationally it is prohibitively expensive if hundreds of pulses need to be optimized. Second, as the pulse number increases, the optimized result becomes more sensitive to the initial guess. Mathematically it is also a challenging question to find the global minimum in a high-dimensional parameter space. For $n+m = 24$, we have considered two initial guesses, namely, an equally spaced DD sequence and a nested-

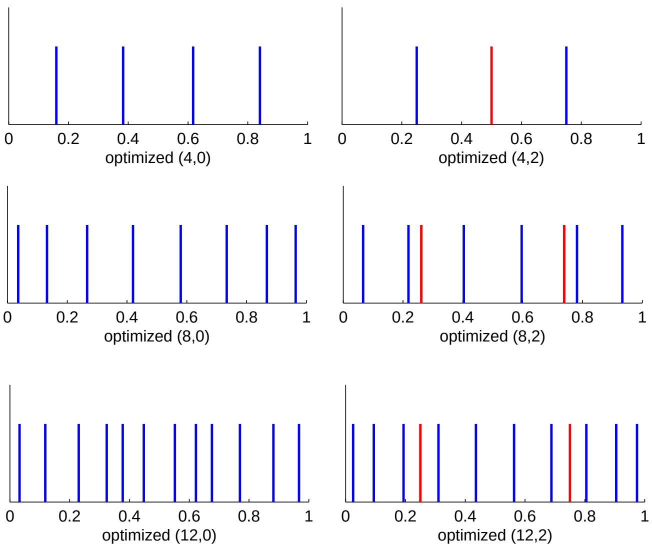

FIG. 2: Optimized pulse timings for $S_1 = 10\Theta(10-\omega), S_2 = 0.1\Theta(0.1-\omega), S_3 = 0.05\Theta(0.05-\omega)$. The left panels are the results if no pulse is applied to the second qubit. The right panels depict the optimized DD sequences if two pulses are applied to the second-qubit pulses ($m = 2$). The grey (red) lines stand for timings of the pulses applied to the second qubit. From top to bottom, $n + m = 4, 8$, and 12. Note that in the upper right panel, the two pulses applied to the second qubit coincide and hence cancel each other.

UDD(4) sequence. The associated results are shown in Table VI. It is seen that for the same noise spectrum, the performance is only slightly better than that for $n + m = 15$. It is hence possible that our result as a locally optimized result is still far away from the globally optimized result. More effective optimization algorithms together with more initial guesses may be the ultimate solution.

It should be also noted that the total pulse number is physically limited. Realistic pulses are imperfect and an ideal instantaneous π rotation is never possible [33]. As a result, the pulse-to-pulse errors may accumulate with increase of the pulse number [34, 35] and hence more pulses do not necessarily lead to better performance. Furthermore, constraint on the minimal pulse interval might place another intrinsic limit on the pulse number we could consider [36]. For fixed total evolution time t, such a constraint imposes an upper bound for the pulse number. For a varying t, the optimized performance in practice will be connected with the available pulsing rate as well as the spectral bandwidth [37, 38].

V. CONCLUSIONS

Using a pure dephasing model incorporating both local and nonlocal Gaussian noise, we have studied how the DD protection of two-qubit states can be better achieved by optimizing the pulse locations as well as the partition of the pulse numbers allocated to each qubit. Compared with nested-UDD as a general-purpose DD scheme for two-qubit systems, our optimization procedure may improve the performance by many orders of magnitude, using only a few tens of instantaneous π pulses. This makes it possible to use much less pulses to obtain a given fidelity of decoherence control in two-qubit systems. The price of this performance gain is the required knowledge of the noise spectrum. In addition, it is also seen that if the noise spectrum decays to zero slowly (e.g., a Lorentzian shape), then even our optimized DD may not perform very well and it is only slightly better than nested-UDD.

TABLE VI: Comparison between optimized two-qubit DD with $n + m = 24$ and nested-UDD(4). The meaning of the table columns are the same as in Table II. Noise spectrum is similar to that considered in Tables II and III.

$S_1 = \omega\Theta(1-\omega), S_2 = \omega\Theta(1-\omega), S_3 = 0.5\omega\Theta(0.5-\omega)$		
Pulse combination	Pulse location	Performance
nested-UDD(4)	5,10	5.21×10^{-9}
C_{24}^{4}	2,9	2.81×10^{-10}
C_{24}^{8}	1,3,5,11	3.31×10^{-11}
C_{24}^{12}	2,4,7,8,10,12	2.34×10^{-11}
$S_1 = \omega\Theta(5-\omega), S_2 = \omega\Theta(5-\omega), S_3 = \omega\Theta(3-\omega)$		
Pulse combination	Pulse location	Performance
nested-UDD(4)	5,10	3.31×10^{-2}
C_{24}^{4}	3,9	1.42×10^{-3}
C_{24}^{8}	1,3,5,10	1.51×10^{-7}
C_{24}^{12}	2,4,6,9,11,12	1.35×10^{-7}

The results here also help to gain more insights into the issue of DD protection in two-qubit or multi-qubit systems. The importance of fighting against the nonlocal noise in two-qubit decoherence control (or more generally, the importance of taking into account the impact of the dynamics of one qubit on the environment of the other qubit) becomes clearer. Without the nonlocal noise, our optimized DD sequence simply degenerates into two single-qubit optimized DD sequences. In the presence of nonlocal noise, the optimized two-qubit DD sequence is quite different from two optimized single-qubit DD sequences, because the pulse numbers and pulse locations for each qubit should be adjusted (in a somewhat subtle manner) based on both nonlocal and local noise. If the local noise spectrum of the two qubits is highly unbalanced, then most pulses should be applied to one of the two qubits to defeat its rapid dephasing. It is these specific strategies, which are not exploited by the rather universal nested-UDD scheme, that makes optimization possible. Note also that in nested-UDD schemes, the number of control pulses or the control order may be different for different control layers. But as seen from

one optimization result in Sec. IV-C, the performance of such type of asymmetric-nested-UDD is still far from optimized.

There have been wide experimental interests in both fundamental aspects of decoherence and the DD approach to decoherence suppression. Two types of experiments may be motivated by this work. First, for two-qubit systems with known noise spectrum (local and nonlocal), it is of immediate interest to apply optimized DD sequences to extend the lifetime of entangled states, hopefully with better efficiency. Second, one may experimentally study the nonlocal noise of a two-qubit system by use of single-qubit DD. That is, if two single-qubit DD sequences are applied on top of each other, then (assuming that the local noise of the two qubits are independent) only the nonlocal noise is not suppressed and hence its impact on two-qubit dephasing may be directly observed. Such type of experiments are of importance to the design of optimized two-qubit DD sequences and to the understanding of decoherence mechanism in two-qubit systems embedded in a solid-state environment.

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